Fluid Mechanics

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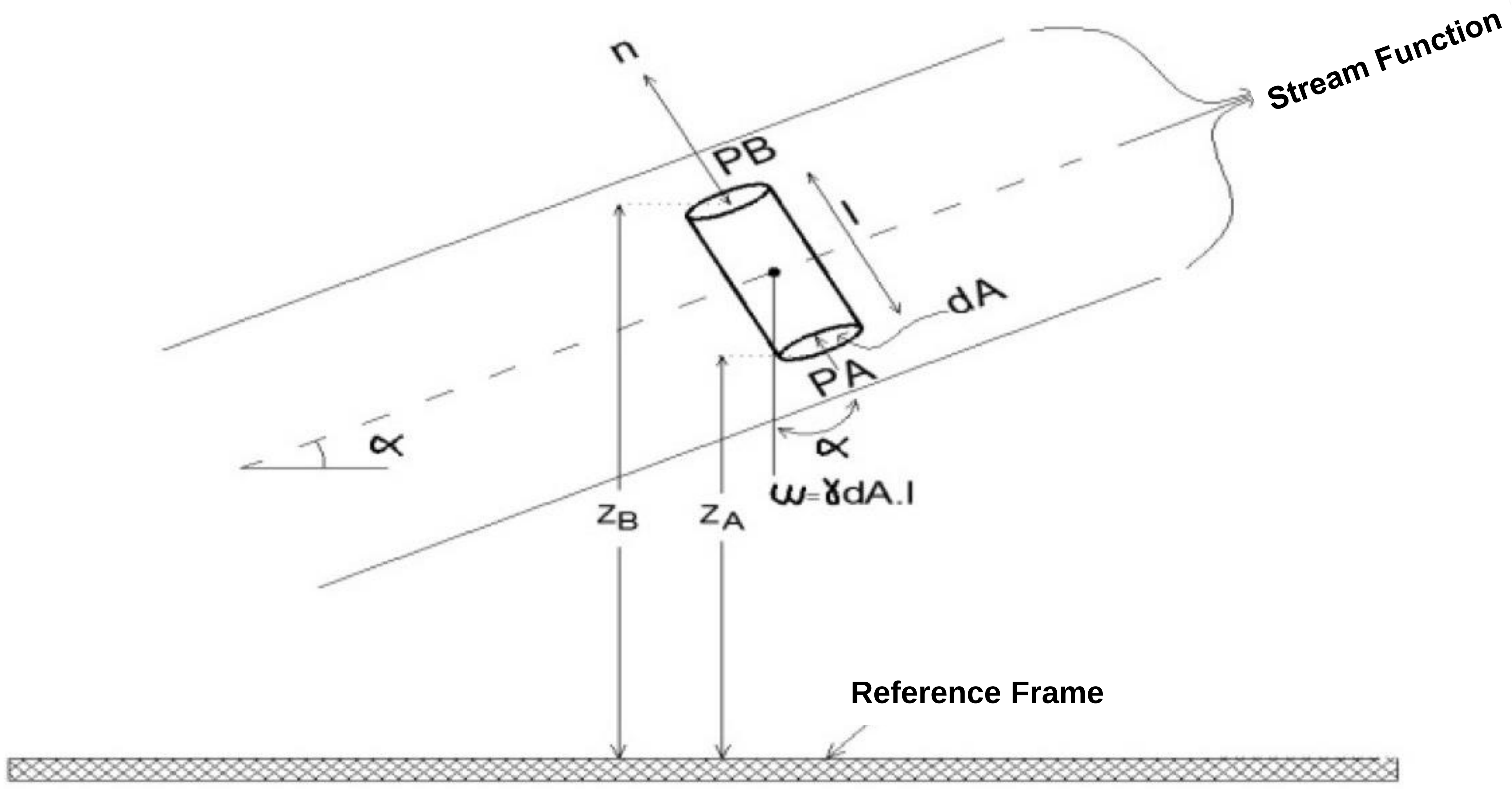
Two Dimensional Flows of Ideal Fluids

Basic equations

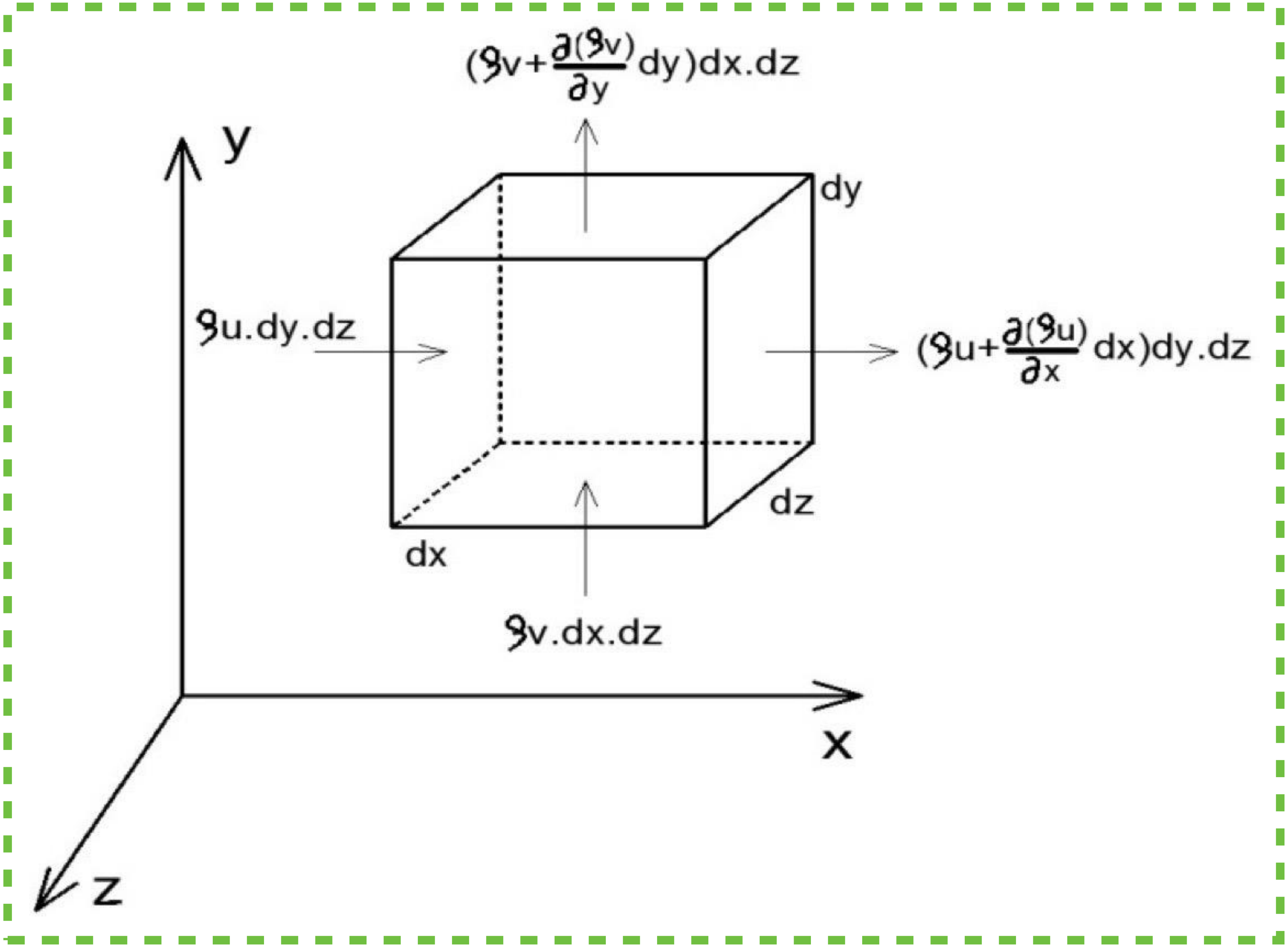
Continuity equation:

Assuming that the ***2-D flow*** be on the ***x-y plane*** and let the flow be *steady*

We can derive the ***continuity equation*** of ***2-D flows*** applying principles of ***conservation of mass*** flow rate on the control volume given in the figure on the next slide


n
Stream Function
PB
l
dA
PA
∝
∝
ω=γdA.l
zB
zA
Reference Frame

The product $\rho . u$ gives us the mass that passes through dz.dy plane in the x-direction per unit time



The same principle can be used to determine the mass flow rate on the other plane

The law of conservation of mass is given as

$$\begin{bmatrix}\text{Change of mass in a} \\ \text{certain control} \\ \text{volume in } \Delta t\end{bmatrix} = \begin{bmatrix}\text{Mass entering the} \\ \text{control volume in } \Delta t\end{bmatrix} - \begin{bmatrix}\text{Mass leaving the} \\ \text{control volume in } \Delta t\end{bmatrix}$$

General continuity equation for ***2-D flows of compressible fluids***

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

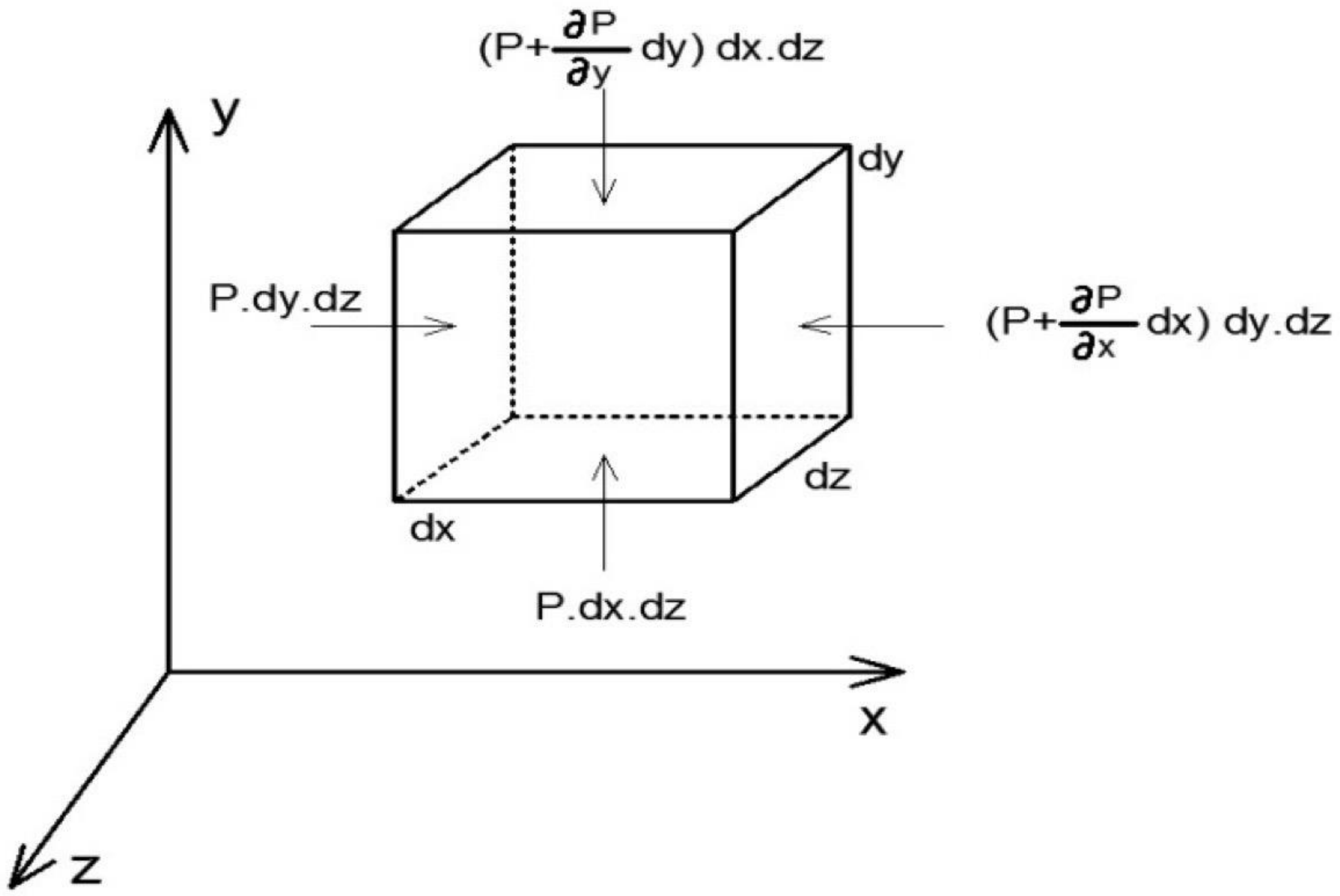
For ***incompressible*** fluids, i.e. is ***constant***

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

For a flow to be ***physically true***, it should **satisfy the continuity equation** which is developed based on the **law of conservation of mass flow rate**

Equation of Motion

Let's have a control volume depicted in the figure



Let the components of the volumetric force (weight) of a unit mass be **X**, **Y** and **Z**, acting in the x, y and z directions, respectively

$$\sum \vec{F}_x = m.a_x \rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} + X = a_x$$

In the same manner

$$\sum \vec{F}_y = m.a_y \rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y = a_y$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+X=a_x$$

and

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}+Y=a_y$$

termed as 2-D Euler's Equations of Motion

We know that

$$a_x=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} \quad \text{and} \quad a_y=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$$

Therefore

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+X=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} \quad \text{and} \quad -\frac{1}{\rho}\frac{\partial p}{\partial y}+Y=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$$

If we assume that the ***weight*** (volumetric force) is acting ***vertically*** in the ***y-direction***

$$a_x = 0$$

Therefore, we will only have Euler's equation in ***y-direction***

If the fluid is ***stagnant***, the velocity components will be ***zero***

$$Y = -g$$

After some arrangements *we end up with*

$$P = -\gamma y + C$$

This is hydrostatic equation derived from Euler's equation of motion

We can develop ***Euler's equations of motion*** for ***3-D*** flows using the same procedures we used previously

Here under are the 3-D equations

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+X=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}+Y=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial y}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}+Z=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial y}$$

2-D Euler's equations of motion for **steady** (permenant) flows

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+X=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} \text{ and } -\frac{1}{\rho}\frac{\partial p}{\partial y}+Y=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$$

After manipulating our equation somehowby ***dx*** *and* ***dy*** *and recalling the notion of* ***streamlines***

$$\frac{u}{dx}=\frac{v}{dy}$$

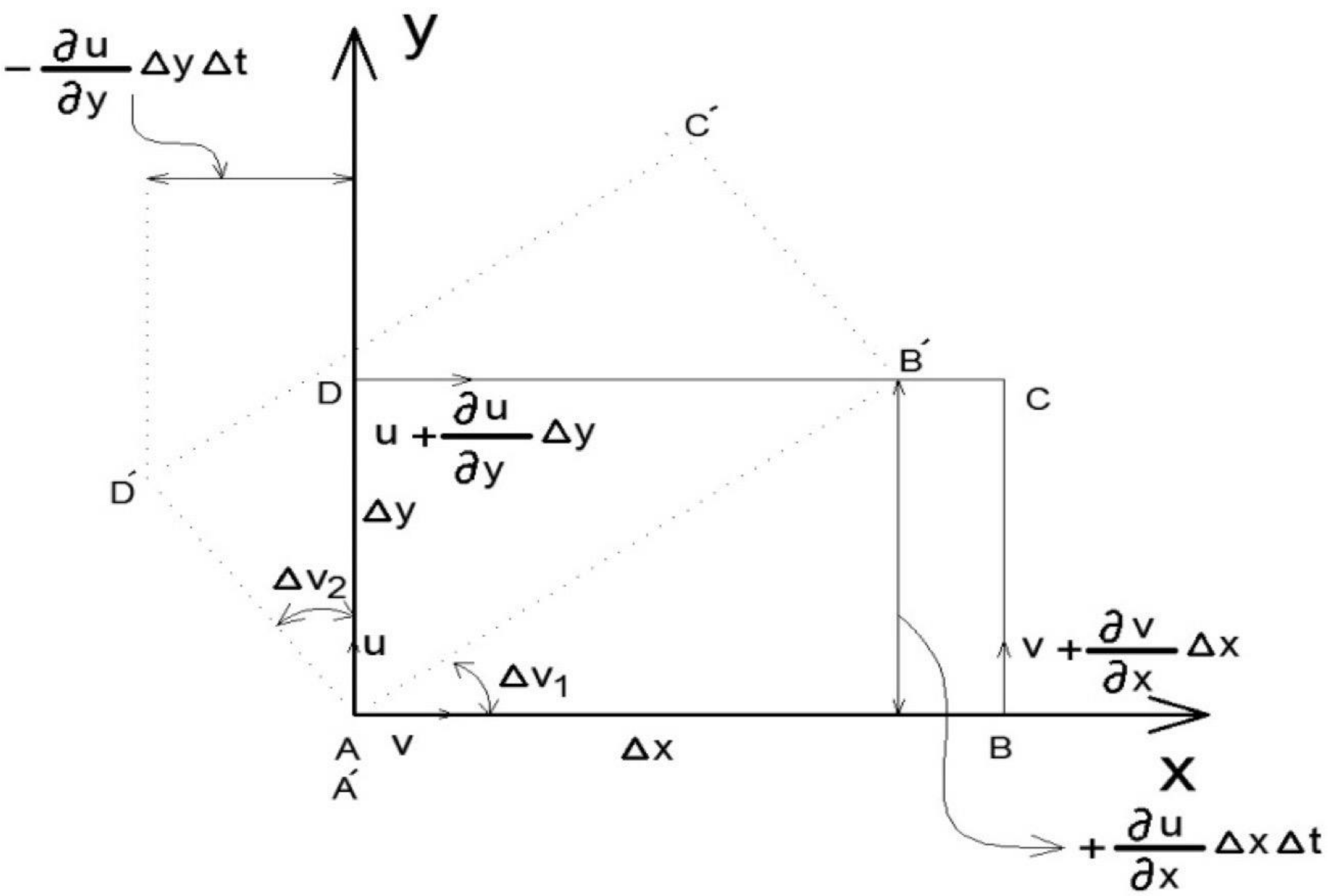
assuming that the volumetric force acts vertically in **y-direction**

We end up with ?

$$Y_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} = Y_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

This is nothing but Bernoulli's equation

The velocity components will have **new values** given below after the element travels **a small distance** within a certain time

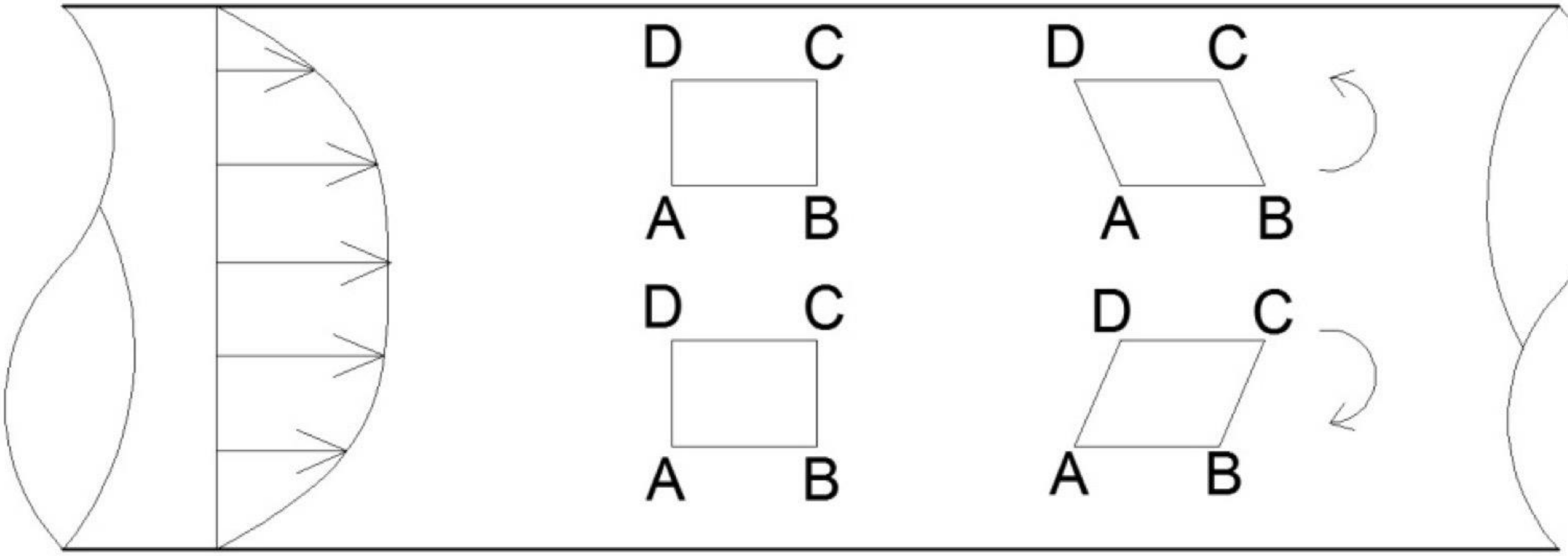


Rotation and Circulation

Rotation

Let's consider a fluid element in a flow

The ***components*** of the ***velocity*** of the element can be represented by ***u*** and ***v***



u will have a value of $u + \frac{\partial u}{\partial y} dy$

v will have a value of $v + \frac{\partial v}{\partial x} dx$

***The angular velocity* is the turning motion that a fluid element makes which is called**

rotation

Let’s take point A as pivot point.

The rotation w about point **A** is therefore given by

$$w_z = \frac{w_{AB} + w_{AD}}{2}$$

$$w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

In the same manner,

$$w_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

and

$$w_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

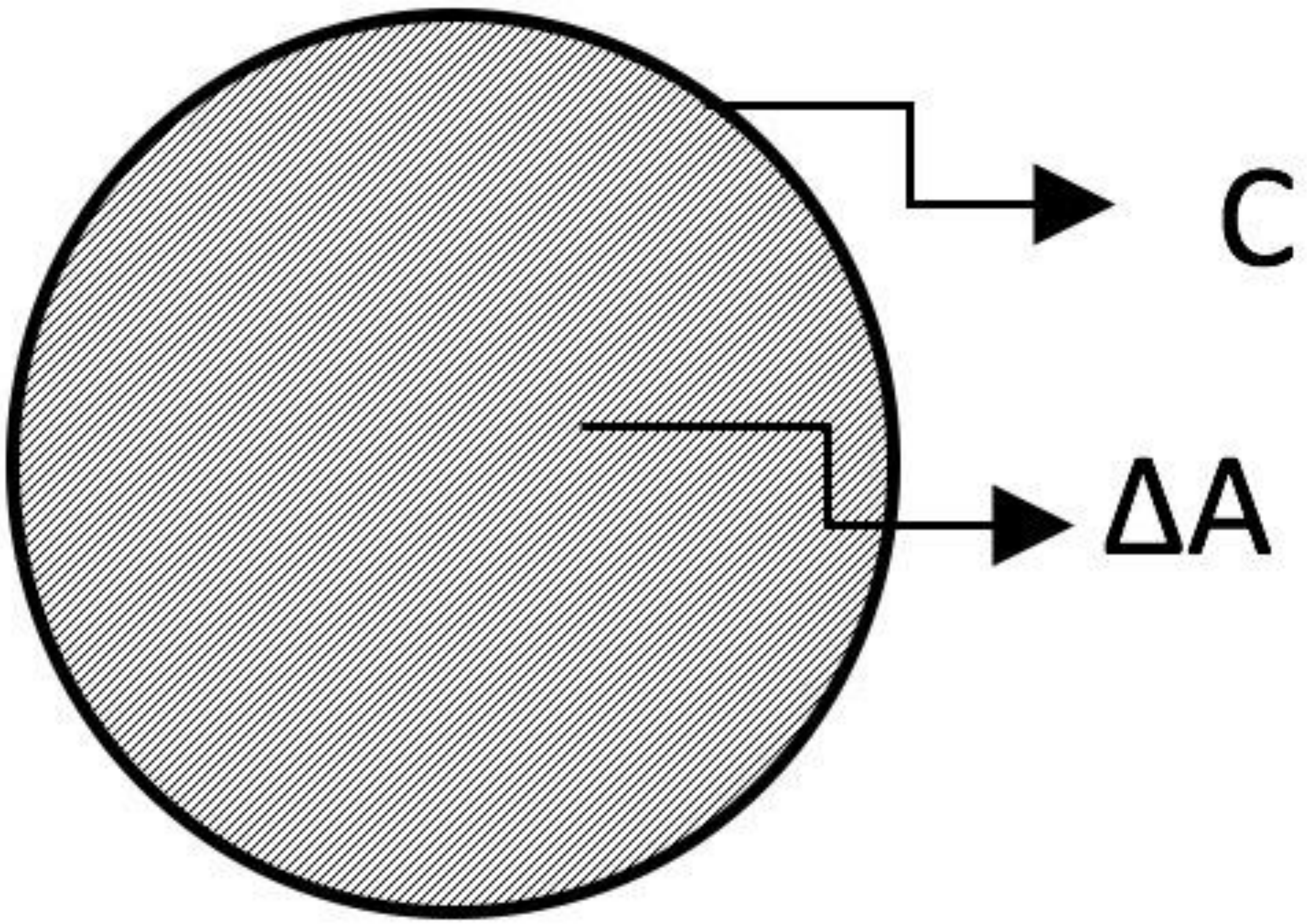
If $w_x = 0$, $w_y = 0$ and $w_z = 0$, the flow is irrotational (potential) flow

If $w_i \neq 0$ where $(i = x, y, z)$, the flow is rotational (non-potential) flow

Circulation

Let's have a closed curve, C

The integral of the velocity along the closed curve is called ***circulation***



If we take a differential distance ds along this curve,
its circulation is given as

$$\Gamma = \oint_C \vec{V} ds$$

Similarity between the concepts
of ***energy*** and ***circulation***

As ***energy*** is the integral force along a certain distance

Circulation is the integral velocity along a certain closed distance.

The circulation equation can be rewritten as follows for *2-D flows*

$$\Gamma = \oint_C (udx + vdy)$$

If we divide the above equation by area on the curved surface the rotation can be found.

$$w = \frac{\Gamma}{A}$$

Stream Function

Flow function is a function where $\psi = \psi(x, y)$

In all physically possible flows

u is the derivative of ψ with respect to y

and

v is the derivative of ψ with respect to x

Therefore, ***flow function*** is a function which satisfies these ***conditions***

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

For this definition to be true in reality the ***continuity equation*** should be satisfied

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ in the continuity equation

$$0 + 0 = 0$$

Therefore ***continuity equation*** is satisfied

If the flow is ***irrotational***

$$w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$$

This shows that the flow function satisfies the ***Laplace equation*** for *irrotational* flow conditions

If we consider a point in a certain flow

the value of ψ at that point gives the values of unit discharge

that passes between the point and the origin

We know that

$$q = \int_0^A u\,dy - v\,dx$$

$$q = \int_0^A d\psi = \psi_A - \psi_o$$

This is the reason why the function is called ***stream function***

Thanks to

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