



Fluid Mechanics
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Potential Function

Potential function is a function where

$$\phi = \phi(x, y)$$

It is a function which satisfies

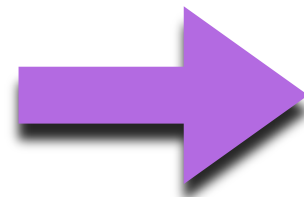
$$u = \frac{\partial \phi}{\partial x}$$

and

$$v = \frac{\partial \phi}{\partial y}$$

This function is only available in **irrotational flows**

Irrotational flow



$$w = \frac{1}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = 0$$

Substituting

$$u = \frac{\partial \phi}{\partial x}$$

and

$$v = \frac{\partial \phi}{\partial y}$$

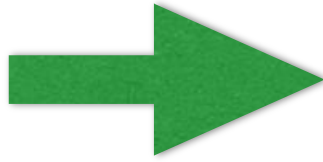
in this equation

$$\left(\frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) \right) = 0$$

$$0 - 0 = 0$$

Therefore, the potential function satisfies the **irrotational flow** condition

Since



$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y}$$

$$\vec{V}_s = \frac{\partial \phi}{\partial s}$$

It has seen that the velocity components are derived from the potential function.

Because of this in irrotational flows, velocity potential motions are called potential flows in short

*In all physically **possible** flows*

u is the derivative of ψ with respect to y

v is the derivative of ψ with respect to x

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}$$

For this definition to be true in reality the **continuity equation** should be satisfied

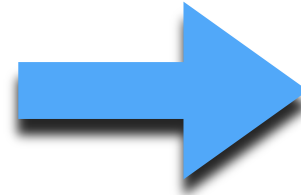
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

*Along the **same flow line**, the flow function has the **same value***

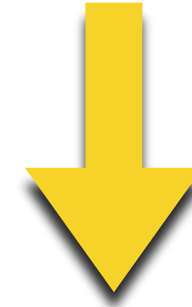
In reality

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$



$$u dy - v dx = 0$$



$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$



$$d\psi = 0$$



$$\psi = \text{Constant}$$

So by equating the flow function to different constant values

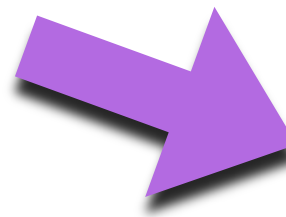
We can prepare **streamlines**
by drawing the **lines graphically**

*In potential flow the **circulation** which is defined as the integral velocity along a closed curve should be **zero**.*

We know that



$$\Gamma = \oint_C \vec{V} ds$$



$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

Substituting

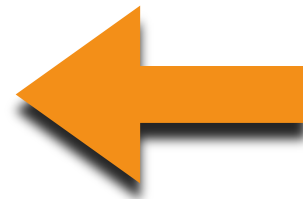
$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

into

$$\Gamma = \oint_C (u dx + v dy)$$



$$\Gamma = \oint_C d\phi$$



$$\Gamma = \oint_C \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right)$$

Therefore

$$\Gamma = \phi \Big|_c^c$$

$$\Gamma = \phi_c - \phi_c = 0$$

Basic Equations of Potential Flows

*The continuity equation and equation of motion developed for 2-D ideal flows are the **same** with in **potential flow conditions***

But we will treat the **energy equation** differently here

For a 2-D flow of x-y plane the two components of Euler's equations of motion were developed as

Adding these two equations we will get

$$-\frac{1}{\rho} \left(\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \right) - g \cdot dy = u \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + v \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right)$$

By using the **exact differential** notion **calculus** and **algebra** we end up with

$$Y_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} = Y_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g}$$

This equation is Bernoulli's equation. It is the energy equation for ideal and permanent flows

If we check the **irrotationality** conditions for potential flows

$$w = \frac{1}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = 0$$

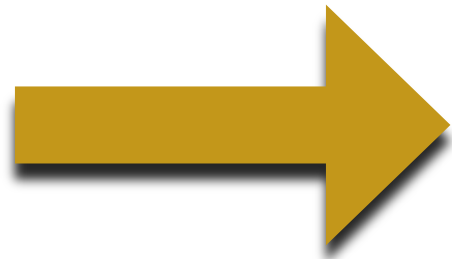
and

Substituting

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}$$

into the equation

yields



$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$$

This shows that the flow function satisfies the Laplace equation in irrotational flow conditions

In *potential flows*, the *potential function* is given as

$$u = \frac{\partial \phi}{\partial x}$$

and

$$v = \frac{\partial \phi}{\partial y}$$

The **continuity equation** for 2D flows is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting **velocity definitions** in the continuity equation

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} = 0$$

This shows that the *potential function* satisfies the Laplace equation

In a certain potential flow

$$\psi = \psi_i = C \quad \text{and} \quad \phi = \phi_i = C$$

The **streamlines** and the **iso-potential** lines are **orthogonal** to each other

If we draw these **orthogonal** lines, there will be a net formed

We call this **net** created as a result of streamlines and iso-potential lines, which are crossing each other perpendicularly a **Flow Net**

Since

$$\psi = \psi_i = \text{Constant}$$



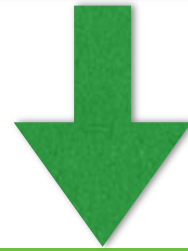
$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$



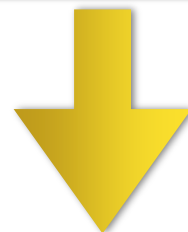
$$\frac{dy}{dx} = \frac{v}{u} = m_1$$

In the same manner

$$\phi = \phi_j = \text{Constant}$$



$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$



$$\frac{dy}{dx} = \frac{-u}{v} = m_2$$

Looking at m_1 and m_2

We can understand that the **streamlines** and the **iso-potential** lines are **orthogonal** to each other

By using stream function velocity potential or both

*We can determine velocity pressure
etc....**at any point** in the flow*

*Problems related to potential flows can be solved
graphically using this flow net*

*For example we can analyze the nature of
infiltrated water under dams, etc...*

Two Dimensional Flow of Real Fluids

Basic Equations

Continuity Equation

The continuity equation for 2-D real flows is given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

If ρ is constant the continuity equation will have the following form

$$\frac{\partial(u)}{\partial x} + \frac{\partial(v)}{\partial y} = 0$$

Equation of motion

Since the flow is real flow viscous forces are present

In real flow the equation of motions in two directions (x and y) are

In the X-direction

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} + X + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$$

In the Y-direction

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} + Y + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}$$

This equation is called Navier-Stokes Equation

If we assume the volumetric force to act only in one direction i.e. vertically and solve the *Navier-Stokes equation*

we will get

$$Z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + h_l$$

h_l is the head loss as a result of frictional (viscous) forces

In the presence of h_l the equation is termed as Energy equation

In the absence of h_l the equation is called Bernoulli's equation

Navier-Stokes equation can be used to solve problems related to 2-D real fluids

However the theoretical solution of Navier-Stokes equation is possible in some special conditions

Since the flow condition is hydrostatic, all velocity components are zero

By solving Navier-Stokes equations we end up with

$$P = -\gamma y + C$$

We have to determine the value of C by taking boundary conditions in to consideration

In conditions when we have infinitely large dimensions, the flow between two plates is considered to be uniform and ***permanent*** and therefore ***laminar***

In this condition, the *Navier-Stokes equations* for 2-D flows are written as follows

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} + X + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} + Y + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}$$

If we take only the component of the equation in the X-direction, the component in the direction where the weight is not action for the given *uniform* and *permanent* flow

we end up with

$$u = \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot \frac{y^2}{2} + C_1 y + C_2$$

We have to use boundary conditions to find C1 and C2

$$u = \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot \frac{y^2}{2} + y \left(\frac{u_0}{a} - \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot \frac{a}{2} \right)$$

This is the **parabolic velocity distribution** equation developed using the *Navier-Stokes equation*

Note: $\frac{\partial p}{\partial x}$

is always negative. As a result, for the velocity to be positive, we have to add a negative sign in front of $\frac{\partial p}{\partial x}$

Thanks to

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