Fluid Mechanics

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Potential Function

Potential function is a function where

$$\phi = \phi(x, y)$$

It is a function which satisfies

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}$$

This function is only available in *irrotational flows*

Irrotational flow →

$$w = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

Substituting $$u = \frac{\partial \phi}{\partial x}$$ ***and*** $$v = \frac{\partial \phi}{\partial y}$$ ***in this equation***

$$\left(\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} \right) \right) = 0$$

$$0 - 0 = 0$$

Therefore, the potential function satisfies the irrotational flow condition

Since →

$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y}$$

$$\vec{V}_s = \frac{\partial \phi}{\partial s}$$

It has seen that the velocity components are derived from the potential function

Because of this in irrotational flows, velocity potential motions are called potential flows in short

In all physically possible flows

u is the derivative of ψ with respect to y

v is the derivative of ψ with respect to x

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}$$

For this definition to be true in reality the *continuity equation* should be satisfied

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Along the same flow line, the flow function has the same value

In reality

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$udy - vdx = 0$$

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$d\psi = 0$$

$$\psi = \text{Constant}$$

So by equating the flow function to different constant values

We can prepare streamlines by drawing the lines graphically

In potential flow the circulation which is defined as the integral velocity along a closed curve should be zero.

We know that

$$\Gamma = \oint_C \vec{V} ds$$

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$$

Substituting $u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}$ **into** $\Gamma = \oint_C (udx + vdy)$

$$\Gamma = \oint_C \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right)$$

$$\Gamma = \oint_C d\phi$$

Therefore $\Gamma = \phi \updownarrow_c^c$ $\quad \Gamma = \phi_c - \phi_c = 0$

Basic Equations of Potential Flows

The continuity equation and equation of motion developed for 2-D ideal flows are the same with in potential flow conditions

But we will treat the *energy equation* differently here

For a 2-D flow of x-y plane the two components of *Euler's equations* of motion were developed as

Adding these two equations we will get

$$-\frac{1}{\rho}(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy)-g.dy=u(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy)+v(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy)$$

By using the *exact differential* notion *calculus* and *algebra* we end up with

$$Y_1+\frac{p_1}{\gamma}+\frac{v_1^2}{2g}=Y_2+\frac{p_2}{\gamma}+\frac{v_2^2}{2g}$$

This equation is Bernoulli's equation. It is the *energy equation for ideal and permanent flows*

If we check the irrotationality conditions for potential flows

$$w = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

and

Substituting $u = \frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$ **into the equation**

yields →

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$$

This shows that the flow function satisfies the *Laplace equation* in *irrotational* flow conditions

In potential flows, the potential function is given as

$$u = \frac{\partial \phi}{\partial x}$$

and

$$v = \frac{\partial \phi}{\partial y}$$

The **continuity equation** for 2D flows is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting velocity definitions in the continuity equation

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} = 0$$

This shows that the *potential function* satisfies the *Laplace equation*

In a certain potential flow

$$\psi = \psi_i = C \quad \text{and} \quad \phi = \phi_i = C$$

The *streamlines* and the *iso-potential* lines are *orthogonal* to each other

If we draw these *orthogonal* lines, there will be a net formed

We call this *net* created as a result of streamlines and iso-potential lines, which are crossing each other perpendicularly a *Flow Net*

Since

$$\psi = \psi_i = \text{Constant}$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$\frac{dy}{dx} = \frac{v}{u} = m_1$$

In the same manner

$$\phi = \phi_j = \text{Constant}$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

$$\frac{dy}{dx} = \frac{-u}{v} = m_2$$

Looking at m_1 and m_2

We can understand that the *streamlines* and the *iso-potential* lines are *orthogonal* to each other

By using stream function velocity potential or both

We can determine velocity pressure etc…at any point in the flow

Problems related to potential flows can be solved graphically using this flow net

For example we can analyze the nature of infiltrated water under dams, etc…

Two Dimensional Flow of Real Fluids

Basic Equations

Continuity Equation

The continuity equation for 2-D real flows is given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0$$

If ρ is constant the continuity equation will have the following form

$$\frac{\partial (u)}{\partial x} + \frac{\partial (v)}{\partial y} = 0$$

Equation of motion

Since the flow is real flow viscous forces are present

D real flow the *equation of motions* in two directions (x and y) a

In the X-direction

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+X+\vartheta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$$

In the Y-direction

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}+Y+\vartheta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$$

This equation is called *Navier-Stokes Equation*

If we assume the volumetric force to act only in one direction i.e. vertically and solve the *Navier-Stokes equation*

we will get

$$Z_1 + \frac{p_1}{\gamma} + \frac{v_1^2}{2g} = Z_2 + \frac{p_2}{\gamma} + \frac{v_2^2}{2g} + h_l$$

h_l ***is the head loss as a result of frictional (viscous) forces***

In the presence of h_l the equation is termed as Energy equation

In the absence of h_l the equation is called Bernoulli's equation

Navier-Stokes equation can be used to solve problems related to 2-D real fluids

However the theoretical solution of Navier-Stokes equation is possible in some special conditions

Since the flow condition is hydrostatic, all velocity components are zero

By solving Navier-Stokes equations we end up with

$$P = -\gamma\, y + C$$

We have to determine the value of C by taking boundary conditions in to consideration

In conditions when we have infinitely large dimensions, the flow between two plates is considered to be uniform and *permanent* and therefore *laminar*

In this condition, the *Navier-Stokes equations* for 2-D flows are written as follows

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+X+\vartheta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}+Y+\vartheta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$$

If we take only the component of the equation in the X-direction, the component in the direction where the weigh is not action for the given *uniform* and *permanent* flow

we end up with

$$u = \frac{1}{\mu}\frac{\partial p}{\partial x}.\frac{y^2}{2} + C_1 y + C_2$$

We have to use boundary conditions to find C1 and C2

$$u = \frac{1}{\mu}\frac{\partial p}{\partial x}.\frac{y^2}{2} + y\left(\frac{u_0}{a} - \frac{1}{\mu}\frac{\partial p}{\partial x}.\frac{a}{2}\right)$$

This is the parabolic velocity distribution equation developed using the ***Navier-Stokes equation***

Note: $\frac{\partial p}{\partial x}$ is always negative. As a result, for the velocity to be positive, we have to add a negative sign in front of . $\frac{\partial p}{\partial x}$

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